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An ambiguity in one-loop quantum gravity

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Abstract. It is argued that the application of the dimensional regularisation technique to one-loop quantum gravity calculations is ambiguous. However, for the calculation of on-mass-shell S-matrix elements, this ambiguity can be resolved by requiring consistency with results obtained from other regularisation schemes. Some discussion is also given of the implications of this work for recent attempts to use higher derivative Lagrangians to solve the renormalisability problem in quantum gravity.

1. Introduction

The problem of finding a renormalisable theory of gravitation has received much attention in recent years. Some progress was made by 't Hooft and Veltman (1974) who observed that, although gravity may not be strictly renormalisable in the usual sense, it is in fact one-loop finite as far as on-mass-shell S-matrix elements are concerned. The argument went as follows. If one uses the background field method then the one-loop counterterms can only be of the form $\sqrt{-g}\,R^2$, $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. The coefficients in front of these counterterms are in fact irrelevant since the identity†

$$\int \mathrm{d}^4x\,\sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2)=0 \qquad (1.1)$$

reduces the independent counterterms to $\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}\,R^2$. These then both vanish if we impose the classical field equation

$$R_{\mu\nu}=0, \qquad (1.2)$$

which is equivalent to going on mass-shell.

't Hooft and Veltman employed dimensional regularisation. This technique has the merit of leading to Green functions which satisfy the Slavnov identities obtained when the more conventional methods of quantum field theory are employed (Capper *et al* 1973, Capper and Ramón Medrano 1974). However, as previous experience has shown, dimensional regularisation is not simply a matter of substituting n for 4 in the Feynman integrals. For instance, the use of the naive four-dimensional Feynman rules in quantum gravity leads to Green functions with finite parts which violate the Slavnov identities (Capper and Ramón Medrano 1973, 1974). In fact either the graviton

† In spaces with a non-trivial topology (such as a de Sitter space) the identity (1.1) is not valid and there is in any case a counterterm proportional to the Euler number. However, in this paper we only consider spaces for which equation (1.1) *is* valid, at least in four dimensions.

propagators or vertices must contain factors of n. Moreover, the use of four-dimensional counterterms can also give rise to unexpected anomalies and this led to the suggestion (Capper and Duff 1974, 1975) that one should really introduce n-dimensional counterterms. It was further shown in the paper by Deser *et al* (1976) that it is the natural insistence on local (rather than nonlocal) n-dimensional counterterms, even in a non-renormalisable theory, which gives rise to anomalies.

We are therefore led to the conclusion that the dimension of space–time (n) should be used right from the beginning of the calculation as a regulating parameter in the sense of Speer (1968). One should then renormalise by defining the *physical* parameters of a theory (e.g., coupling constant, masses, etc) and finally let n go to 4 right at the end of the calculation. If the resulting Green functions are finite then we have a renormalisable theory. Any less committed attempt to use the dimension of space–time as a regulating parameter will lead to inconsistencies such as incorrect Feynman rules (which in turn lead to incorrect finite parts of scattering amplitudes) as well as problems of how to interpret the scalar product between vectors in multi-loop diagrams.

At first sight it seems straightforward to apply this philosophy to show that quantum gravity is one-loop finite. The counterterms would seem to be of the form

$$\frac{1}{(n-4)}\int \mathrm{d}^n x\,\sqrt{-g}\,(\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}). \tag{1.3}$$

Equation (1.1) presumably generalises to something of the form

$$\int \mathrm{d}^n x\,\sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2)=(n-4)\times(\text{something}), \tag{1.4}$$

which again reduces the counterterms to the form $R_{\mu\nu}R^{\mu\nu}$ and R^2. Equation (1.2) is still the correct n-dimensional field equation and hence reduces the counterterms to zero. The flaw in this argument is that equation (1.4) is incorrect. As we shall see, the object

$$F\equiv\int \mathrm{d}^n x\,\sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2) \tag{1.5}$$

does not go to zero like $(n-4)$.

Unfortunately the standard proofs of the vanishing of F in four dimensions appear to be of no value in discovering the correct form of equation (1.4). However, one could try expanding F in terms of the graviton field $\phi_{\mu\nu}$, using for instance

$$g_{\mu\nu}=\eta_{\mu\nu}+K\phi_{\mu\nu} \tag{1.6}$$

in the hope that factors of $(n-4)$ might occur. In fact, there are not even any factors of n. Equation (1.5) written in terms of $\phi_{\mu\nu}$ can only be seen to vanish in four dimensions by explicitly writing out the fields and derivatives in terms of components and doing integrations by parts. This is actually very messy to accomplish. The reason is that for the bilinear terms the identity (equation (1.1)) *is* true in n dimensions rather than 4. This is due to the fact that only four different indices occur and it is thus impossible to construct terms which are significantly different from those occurring in four dimensions. Hence the first non-trivial case is of third order in $\phi_{\mu\nu}$ and is therefore very complicated. Fortunately, as we show in the next section, there is an equivalent identity to equation (1.1) in two dimensions which serves to illustrate our point.

It is, of course, tempting to first put n equal to 4 in the integral of equation (1.3) and then to take the limit $n\to 4$. However, it is precisely such manipulations which can lead to the erroneous evaluation of anomalies and, moreover, give rise to manifestly incorrect mathematics. The limit of a product is *not* in general the product of the limits. Evaluating equation (1.3) by first putting n equal to 4 in the integration is directly analogous to the following (incorrect) series of manipulations:

$$\lim_{n\to 4}\left(\frac{1}{(n-4)}(5+6+7+\ldots+n)\right)=\left(\lim_{n\to 4}\frac{1}{(n-4)}\right)\left(\lim_{n\to 4}(5+6+7+\ldots+n)\right)$$
$$=\lim_{n\to 4}\frac{1}{(n-4)}\times \text{zero}=0. \tag{1.7}$$

It is therefore important to attempt to find out in what manner the expression in equation (1.8) vanishes as n goes to 4.

2. The two-dimensional Gauss–Bonnet identity

Analogous identities to equation (1.1) exist for any even-dimensional space–time (with a trivial topology), and they are known as Gauss–Bonnet identities. The one appropriate to two dimensions is

$$\int \mathrm{d}^2 \times \sqrt{-g}\, R = 0\,. \tag{2.1}$$

It is straightforward to show the validity if equation (2.1) and, for instance, the reader can find a simple proof in the paper by 't Hooft and Veltman (1974). However, such proofs are limited to two dimensions and as such are insufficient for our purposes. What we need to know is how the expression

$$Y=\int \mathrm{d}^n x\, \sqrt{-g}\, R \tag{2.2}$$

goes to zero as n goes to two. One possible way of attempting to do this is by using equation (1.6). After integration by parts, we obtain to second order in $\phi_{\mu\nu}$

$$Y=K^2\int \mathrm{d}^n x\, \left(\tfrac{1}{4}\phi_{\alpha\alpha}\phi_{\rho\rho,\mu\mu}-\tfrac{1}{4}\phi_{\rho\sigma}\phi_{\rho\sigma,\mu\mu}-\tfrac{1}{2}\phi_{\alpha\alpha}\phi_{\rho\mu,\rho\mu}+\tfrac{1}{2}\phi_{\rho\sigma}\phi_{\sigma\mu,\mu\rho}\right). \tag{2.3}$$

The expression for Y is not obviously zero even in two dimensions, so we now examine particular combinations of components†.

The $\phi_{11}\phi_{11}$ contribution to Y is of the form

$$\int \mathrm{d}^n x\, \phi_{11}\left(\tfrac{1}{4}\partial^2-\tfrac{1}{4}\partial^2-\tfrac{1}{2}\partial_1^2+\partial_1^2\right)\phi_{11}. \tag{2.4}$$

The $\phi_{11}\phi_{22}$ contribution to Y is of the form

$$\int \mathrm{d}^n x\, \phi_{11}\left(\tfrac{1}{4}\partial^2+\tfrac{1}{4}\partial^2-\tfrac{1}{2}\partial_2^2-\tfrac{1}{2}\partial_1^2\right)\phi_{22}=\tfrac{1}{2}\,\mathrm{d}^n x\, \phi_{11}\left(\partial_3^2+\partial_4^2+\ldots+\partial_n^2\right)\phi_{22}. \tag{2.5}$$

† For simplicity, we use the notation $\partial^2=\partial_\mu\partial^\mu$ and a $++++$ metric.

The $\phi_{12}\phi_{12}$ contribution to Y is of the form

$$\int \mathrm{d}^n x\, \phi_{12}(-\tfrac{1}{4}\partial^2 - \tfrac{1}{4}\partial^2 + \tfrac{1}{2}\partial_1^2 + \tfrac{1}{2}\partial_2^2)\phi_{12} = -\tfrac{1}{2}\int \mathrm{d}^n x\, \phi_{12}(\partial_3^2 + \partial_4^2 + \ldots + \partial_n^2)\phi_{12}. \tag{2.6}$$

Finally, the $\phi_{11}\phi_{12}$ contribution to Y is of the form

$$\int \mathrm{d}^n x\, \phi_{11}(-\tfrac{1}{2}\partial_1\partial_2 - \tfrac{1}{2}\partial_2\partial_1 + \tfrac{1}{2}\partial_1\partial_2 + \tfrac{1}{2}\partial_2\partial_1). \tag{2.7}$$

As can be seen, all the contributions vanish for two dimensions but only those of equations (2.4) and (2.7) vanish for n dimensions. The remaining contribution to Y is a chain of terms which contracts to zero in number as n goes to two. There is no $(n-2)$ factor in Y and, since the chain of terms which appears in Y consists of different components of n-vectors, it is impossible to sum them to get an $(n-2)$ factor†. The vanishing of Y as n goes to 2 is thus a purely discrete effect, which actually is not all that surprising considering the topological origin of the Gauss–Bonnet formulae.

A similar result presumably holds for the terms cubic in $\phi_{\mu\nu}$ for the expression

$$\int \mathrm{d}^n x\, \sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \tag{2.8}$$

Since there are again no $(n-4)$ factors. This remark follows from inspection of the equations for $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$ and R in terms of $\Gamma^{\sigma}_{\rho\lambda}$, $\Gamma^{\sigma}_{\rho\lambda}$ in terms of $g_{\rho\lambda}$ and finally $\phi_{\rho\lambda}$, which reveals that there can never occur a contracted Kronecker delta which is the only way of producing a factor of n (and thus possibly $n-4$).

3. The Gauss–Bonnet identity in terms of tree diagrams

Although equation (1.4) cannot contain a factor of $(n-4)$, it is quite conceivable that scattering amplitudes might, due to the occurrence of Kronecker deltas in both the vertices and propagators.

The ideal approach to our problem would be actually to calculate the one-loop corrections to a non-trivial graviton scattering problem using conventional Feynman diagrams (e.g., two graviton → two graviton scattering) and to investigate if this amplitude really is finite in four dimensions. In practice, owing to the large number of complicated diagrams, this is an enormous calculation which we are at present only contemplating. However, as a preliminary investigation, we might ask whether the addition of a counterterm of the form

$$\frac{\lambda}{n-4}\int \mathrm{d}^n x\, \sqrt{-g}\, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \tag{3.1}$$

† If one had a chain of terms of the form

$$f(n) = 3+4+5+\ldots+n \qquad \text{for } n \geqslant 3$$
$$= 0 \qquad \text{for } n = 2$$

then it would indeed be possible to sum this chain of terms to obtain

$$f(n) = \tfrac{1}{2}(n-2)(n+3) \qquad \text{for } n \geqslant 2$$

and hence a factor of $(n-2)$. However, the chain of terms in equation (2.6), for instance, is not of this form and consists of components of n-vector rather than numbers. It cannot therefore be summed in this way.

contributes zero to the scattering amplitude to first order in λ. We have calculated all the diagrams shown in figure 1 using the computer program SCHOONSCHIP (Strubbe 1974; further details will be given by one of the present authors (Kimber, PhD thesis in preparation)). The results were checked by verifying that the replacement of a polarisation vector ϵ_α by the appropriate external momentum made the total amplitude vanish. Despite the use of n-dimensional propagators and vertices, no factors of $(n-4)$ appeared in the final amplitude. Moreover, the amplitude did not vanish even with all the physical constraints imposed such as massless external gravitons and momentum conservation. For simplicity we then went to the centre-of-mass frame and, motivated by arguments given in § 2, explicitly went to four dimensions and introduced all the individual components of the various four-vectors. The amplitude still did not vanish! In fact, it was only by introducing an explicit set of polarisation vectors and looking at each of the different helicity amplitudes that the amplitude could be shown to vanish in four dimensions. Thus the vanishing of the four-dimensional amplitude is a purely discrete effect, analogous to that shown in § 2, and *no factors of* $(n-4)$ *occur.*

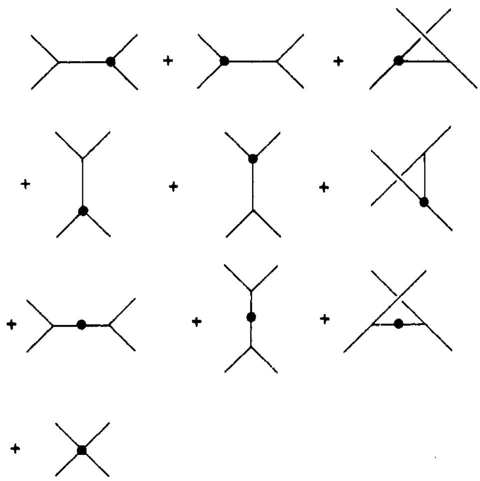

Figure 1. Order-λ contributions of the counterterm $[\lambda/(n-4)]\int d^nx\,\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ to graviton–graviton scattering.

4. Conclusion

Our results appear to show that there is no mathematically well defined value for

$$\lim_{n\to 4}\frac{1}{(n-4)}\int d^nx\,\sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2) \qquad (4.1)$$

either when expressed in terms of the graviton field or S-matrix elements. Although we would hesitate to claim that this shows that quantum gravity is not one-loop finite, one ought to be able to calculate a process such as graviton–graviton scattering in n dimensions using conventional Feynman diagram techniques and the resulting amplitude should be unambiguously finite in the limit $n\to 4$. Although we have not yet carried out this calculation, the results presented here indicate that is unlikely to be true; presumably expressions of the form (4.1) would arise and one would not have a way of handling such terms. However, in order to obtain consistency with other regularisation schemes (for a review, see, for example, De Witt (1975)) one could adopt the following procedure. The scattering amplitude for a process such as graviton–graviton scattering is worked out using the *n-dimensional* vertices and propagators. The external legs are put on-mass-shell and the external momenta and polarisation vectors are restricted to four dimensions. Only *after* this has been done should the internal momenta be restricted to four dimensions. The resulting amplitude should then be finite. However, it must be realised that this method of taking the $n \to 4$ limit is precisely analogous to equation (1.7). Although, in the context of on-mass-shell S-matrix elements, the prescription outlined above does appear to be self-consistent, our results probably have serious consequences for attempts to renormalise gravity by using R^2-type Lagrangians (Stelle 1978, Julve and Tonin 1978, Salam and Strathdee 1978). So far, in such models the $\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term has been omitted from the basic Lagrangian (even in the context of dimensional regularisation) on the grounds that it can be rewritten by means of the Gauss–Bonnet formulae in terms of $\sqrt{-g}\,R^2$, $\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}$ plus additional terms which would only contribute finite amounts to one-loop calculations. However, since Stelle (1977), Julve and Tonin (1978) and Salam and Strathdee (1978) are concerned with the renormalisation of off-mass Green functions (rather than the finiteness of on-mass-shell S-matrix elements) the ansatz outlined above cannot be applied. Thus, unless a different method of resolving the ambiguities discussed in this paper can be found, it would seem likely that all three R^2-type terms are required in the n-dimensional Lagrangian for such models. To verify this would require the evaluation of triangle diagrams rather than the much simpler self-energy.

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